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Gravity currents with variable inflow

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We have performed a series of experiments on two-dimensional gravity currents for which the inflow rate at the origin is a power-law function of time, $t^{\alpha-1}$. The theoretical results of Huppert (1982), for currents in which there is a balance between buoyancy and viscous forces, are found to be valid for a wide range of conditions. A large numbers of experiments at a critical value of $\alpha = \frac{7}{4}$ show very precise agreement with the theory, while values of a parameter that separates regions in which either viscous forces or inertial forces dominate are well within limits one would expect from the order-of-magnitude arguments used.

1. Introduction

Two recent and related papers (Didden & Maxworthy 1982; Huppert 1982) consider primarily the problem of a viscous gravity current for which the inflow rate at the source is constant or formed by the release of a fixed volume of fluid. The theory, presented in the latter paper, considers the flow to be slender so that exact results can be computed by making the usual assumptions of lubrication theory. (See also the first note added in proof in Huppert (1982) for comments on the relationship between his work and that of Barenblatt (1952).) The comparison between the theory and experiments reported in both papers is very satisfying. This second paper also calculates the spreading rate of viscous, two-dimensional and axisymmetric currents for inflow rates that vary as some arbitrary, but fixed, power of the time from the start of the motion. It was shown by Huppert that there exists one critical value of the parameter α that separates two distinctly different types of evolution. Below the parameter's critical value, which includes the constant-flow-rate and fixed-volume-release cases mentioned above, the initial flow is governed by a balance between a driving buoyancy force and a retarding inertial force with a transition to a balance between buoyancy forces and viscous forces at some time after flow initiation. Above the critical parameter value the reverse is true, the flow starts in a viscous–buoyancy balance and undergoes transition to an inertia–buoyancy balance at a later time. At the critical value the flow exists for all time at either one or other of these states, depending on the value of a certain related parameter which measures the relative magnitude of the inertia and viscous forces. These results are sufficiently intriguing to prompt an experimental investigation into their validity.

The flows are of interest to those involved in the study of motions in the natural environment. For example, flows of fresh water from a spring run-off into lakes and fjords rarely take place with a constant flow rate, and the consequent evolution of the intrusions thus formed may be incorrectly estimated by using a constant-flow

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model. A number of similar problems: flash floods, variable stratified flows in the ocean, lava flows from volcanoes, etc. are all related to the simple laboratory model we present here, which represents a first step on the way to a more complete understanding.

2. Theoretical preliminaries

Here we present the appropriate results from Huppert (1982) to be used in what follows. We consider only two-dimensional flows for which the total volume of heavy fluid in the current, per unit width of channel, varies as qt^{α}. For the case of a viscous–buoyancy force balance this theory gives:

$$L = k_{\mathrm{V}}\left\{\frac{g'q^3}{\nu}\right\}^{\frac{1}{5}} t^{\frac{1}{5}(3\alpha+1)}, \tag{1}$$

where L is the total length of the current, g' a reduced gravity equal to $(\Delta\rho/\rho)\,g$, $\Delta\rho$ being the density difference between the current and its surroundings and ρ the density of the intrusion, k_{V} is a calculable constant (related to Huppert's η_{V} by $k_{\mathrm{V}} = \eta_{\mathrm{V}}(\frac{1}{3})^{\frac{1}{5}}$) and ν is the kinematic viscosity of the intrusion. The viscosity of the ambient fluid does not appear, since it is assumed that the viscous stress at the free surface of the current is very much smaller than that due to the friction at the bottom (see Huppert 1982).

The inertia–buoyancy balance gives

$$L = k_{\mathrm{I}}(\alpha)\,\{g'q\}^{\frac{1}{3}}\, t^{\frac{1}{3}(\alpha+2)}, \tag{2}$$

by order-of-magnitude arguments when it is assumed that the motion of the intrusion is self-similar and the flow is not controlled by the non-similar, dynamical processes occuring at the head of the current. The constant $k_{\mathrm{I}}(\alpha)$ can only be found by recourse to experiments.

The transition time at which inertial and viscous forces are equal can be calculated either by equating inertial and viscous forces or by noting that at transition (1) and (2) must be identical (see §4 also), and is given by†

$$t_{\mathrm{TR}} = \left\{\left[\frac{k_{\mathrm{V}}(\alpha)}{k_{\mathrm{I}}(\alpha)}\right]^{15} \frac{q^4}{g'^2\nu^3}\right\}^{1/(7-4\alpha)} \tag{3}$$

The critical value of α mentioned in §1 is then seen to be $\frac{7}{4}$. For $\alpha = \frac{7}{4}$ inertial effects are small for all time if the dimensionless Julian number

$$J = \left\{\frac{k_{\mathrm{I}}(\alpha)}{k_{\mathrm{V}}(\alpha)}\right\}^{15} \frac{\nu^3 g'^2}{q^4} \gg 1.$$

For $\alpha < \frac{7}{4}$, t_{TR} is the time for transition from an inertial to a viscous flow, and for $\alpha > \frac{7}{4}$ it is the time for transition from viscous to inertial flow.

Based on these predictions we designed an experiment to test their validity, the results of which are presented in §4 and discussed in §5. An appendix on the flows created by discontinuous changes in flow rate is presented finally, since they have some bearing on the results presented in the main body of the paper.

† The fact that the transition time is so sensitive to changes in the values of the constants, especially as will be seen to changes in $k_{\mathrm{I}}(\alpha)$, is simply a reflection of the fact that the slopes of the two relationships (1) and (2) are generally very closely matched, and hence a small change in $k_{\mathrm{I}}(\alpha)$ results in an enormous change in t_{TR}.

3. Experimental procedure

The tank used in this study was 3 m long, 50 cm deep and 20 cm wide. It was filled to a depth of 40 cm with fresh water, a small portion of which was withdrawn and mixed with an accurately measured amount of common salt. This ensured that any density difference was only due to the salinity difference and that temperature and possible doubly diffusive effects were avoided (see Maxworthy 1983). This dense fluid was dyed and placed in a reservoir, which in turn was connected to a two-dimensional flow inlet diffuser in the bottom of the tank via a pump, flowmeter and control valve. The flowmeter was a conventional rotameter with a modified scale. Depending on the value of α required, marks were placed at intervals on a strip of paper taped to the flowmeter tube. Each mark represented the required location of the flowmeter bob at a given instant in order to preserve not only the correct value of α but also the correct value of q. In practice this meant that the flow rate had to be changed to a new value in discrete steps. Thus in order to change the value of q for fixed α, experiments were run in which the flow rate was nominally changed to a new value at 5, 10, 15, 20, 30 or 40 s intervals ten times during any one experiment. The latter three cases, involving changes over the larger time intervals, were actually divided into smaller divisions, and, correspondingly, changed more frequently, in order to approach the desired continuous flow-rate change more closely.

The dyed gravity current was photographed against a bright background with a length scale, stopwatch and pointer, to indicate the nose of the current, all within the field of view.

4. Results

Based on the comments in §2 we expect the most interesting behaviour to occur for a value of $\alpha = \frac{7}{4}$, and performed 24 experiments at four different values of $\Delta\rho/\rho$ (0·0013, 0·0038, 0·012 and 0·063) and six values of q (0·0182, 0·0230, 0·0311, 0·0380, 0·0514 and 0·088 $\mathrm{cm^2/s^{\frac{7}{4}}}$).

Two sets of representative results are shown for $\Delta\rho/\rho = 0{\cdot}0013$ and $0{\cdot}012$ in figures 1 and 2. In figure 1 we show curves for six values of q which indicate a transition from a viscous–buoyancy balance to an inertial–buoyancy value at a value of $J = 2{\cdot}9$, i.e. where the slope of the line of L, for fixed t, versus q changes slope from 0·6 to 0·33 (equations (1) and (2)). Note also from both (1) and (2) that the predicted slope of the six lines is 1·25 and that the measured values are consistently lower at 1·16. Similarly in figure 2 we show that the slope of all lines of L *vs.* t is 1·18 and that the viscous–inertial transition takes place at a value of $J = 3{\cdot}5$. For the other cases not presented in detail here, $n = 1{\cdot}20$ and $J = 3{\cdot}4$ for $\Delta\rho/\rho = 0{\cdot}0038$ and $n = 1{\cdot}21$ for $\Delta\rho/\rho = 0{\cdot}063$. No observable transition took place for this later value of α, indicating that J_{TR} was smaller than the smallest value of $J \approx 60$ of this set of experiments. From the three experiments for which a transition was found, the average value of J_{TR} is 3·2.

Equations (1) and (2) have been used to create a scaling that reduces all of these data to two plots. In figure 3 we show the data for those cases for which L, at fixed t, scales as $q^{0{\cdot}6}$, i.e. the viscous cases. We note an interesting result: although all the individual curves had values of n less than the theoretical value of 1·25, when combined they overlap in such a way that the theoretical curve is a remarkably good fit to the data. A value of the constant of 0·756 appears to fit the data slightly better than the theoretical value of 0·734, and we will use the former value in what follows.

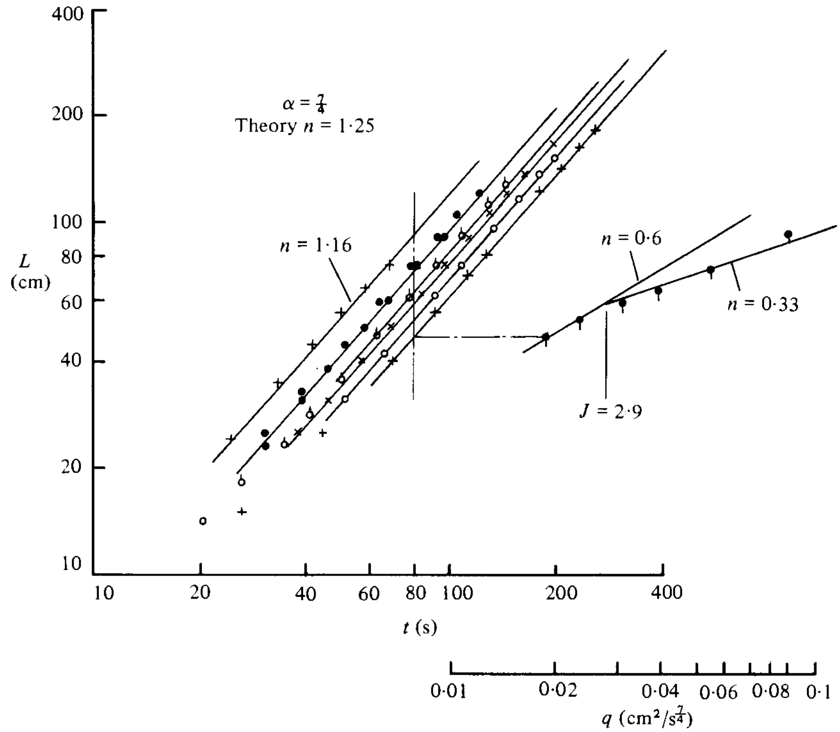

FIGURE 1. Current length L versus time t for $\Delta\rho/\rho = 0{\cdot}0013$, $\alpha = \frac{7}{4}$, and length at a fixed time $L(t_F)$ versus q: —+—, $q = 0{\cdot}0182\ \mathrm{cm^2/s^{\frac{7}{4}}}$; —●—, 0·023, two runs; —ȯ—, 0·0311; —×—, 0·038; —○—, 0·0514; —+—, 0·088. The inserts in the lower right-hand corners of this and figure 2 give the length of the current at a fixed time $L(t_F)$ versus q(—●—) for each case, and is used to determine the value of J_{TR}.

Similarly the inertial cases, i.e. those in which L, at fixed t, scales as $q^{0{\cdot}33}$, are shown in figure 4. Here the data scatter is somewhat greater, but again a slope of $n = 1{\cdot}25$ fits well with a constant $k_I = 0{\cdot}726$.

From (3) an averaged value of the transition number J_{TR} must be unity, with $(k_V(\frac{7}{4})/k_I(\frac{7}{4}))^{15} = 2{\cdot}2$, while, as we have already shown, the value of J_{TR} from the raw data curves is 3·2. However, because $J_{TR} \propto q_{TR}^{-\frac{4}{7}}$ the values of q_{TR} in the latter cases would only need to have been raised by a small amount, approximately 15 %, to bring about agreement between the two results. Thus, while (3) with $(k_V(\frac{7}{4})/k_I(\frac{7}{4}))^{15} = 2{\cdot}2$ gives the best estimate of J_{TR}, in the range $0{\cdot}5 < J < 5$ either (1) or (2) gives an adequate description of the measured spreading relationship, to within approximately 10 %.

We also performed a smaller number of experiments for values of α on either side of the critical value. In figure 5 we show the results of four experiments for which $\alpha = \frac{3}{2}$. In all cases the slope is that of the viscous–buoyancy balance, after an initial slope which is somewhat less than unity. This latter characteristic, which is mainly a consequence of the experimental technique (see §5), is also seen in some of the curves of figures 1 and 2 and will be seen again in the results for other values of α. For example, experiments run at the one other value of α below the critical value are presented in the appendix. For the upper curve of figure 8, $\alpha = 1$ and again the slope changes are from quite small values to the predicted one so that agreement with the theory is excellent and is actually better than that reported in Didden & Maxworthy (1982).

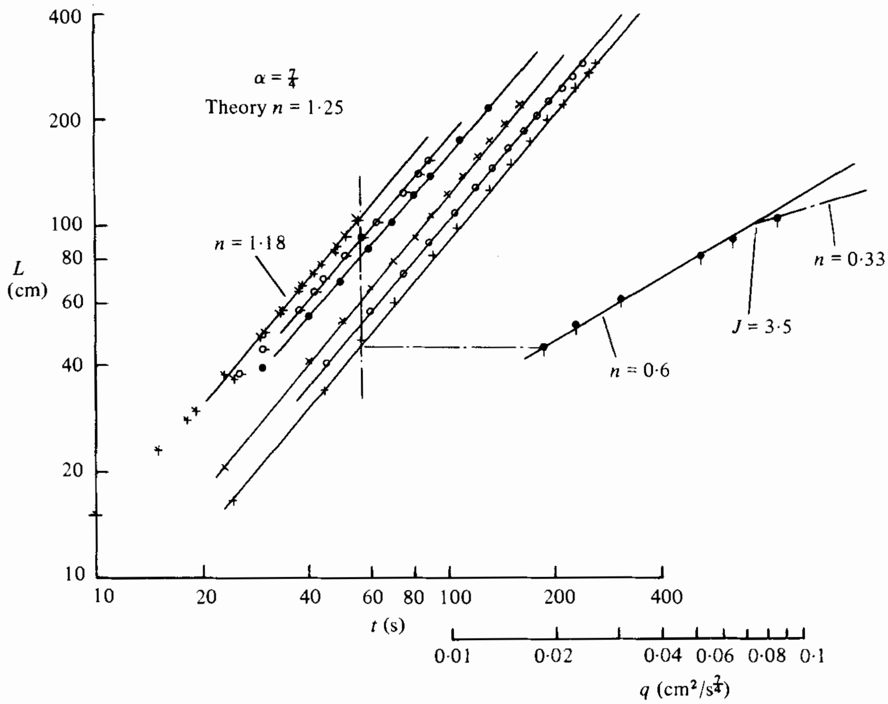

FIGURE 2. Current length L versus time t for $\Delta\rho/\rho = 0{\cdot}012$, $\alpha = \frac{7}{4}$, and length at a fixed time $L(t_F)$ versus q: —✳—, $q = 0{\cdot}0182$ cm²/s$^{7/4}$; —●—, 0·023, two runs; —ó—, 0·0311; —×—, 0·038; —○—, 0·0514; —+—, 0·088; —♦—, $L(t_F)$ versus q.

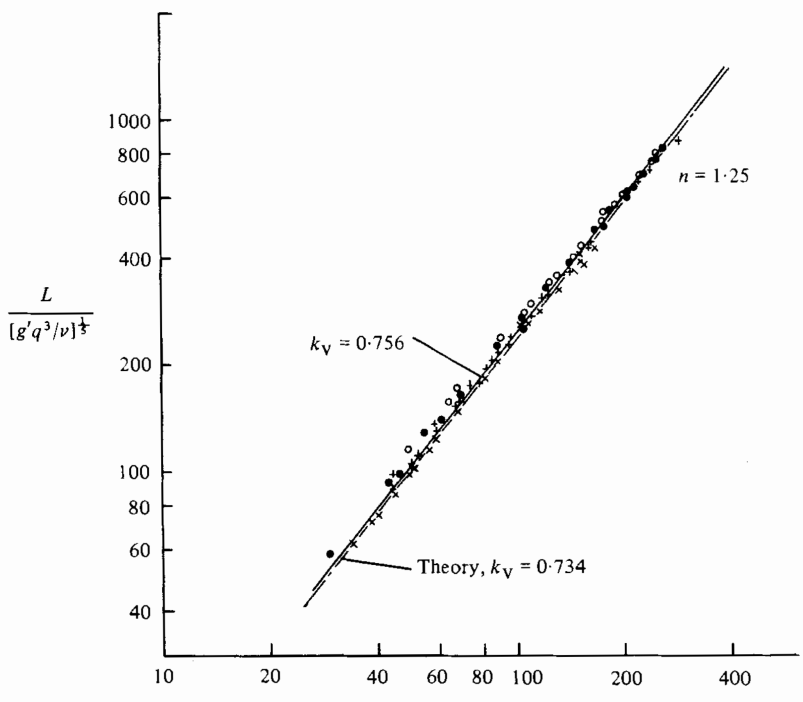

FIGURE 3. $L/[g'q^3/\nu]^{\frac{1}{5}}$ versus time t for buoyancy–viscous currents for six values of q and four values of $\Delta\rho/\rho$: —○—, $\Delta\rho/\rho = 0{\cdot}0013$; —+—, 0·0038; —●—, 0·012; —×—, 0·063. Theoretical curve from Huppert (1982), $L/[g'q^3/\nu]^{\frac{1}{5}} = 0{\cdot}734t^{\frac{5}{4}}$.

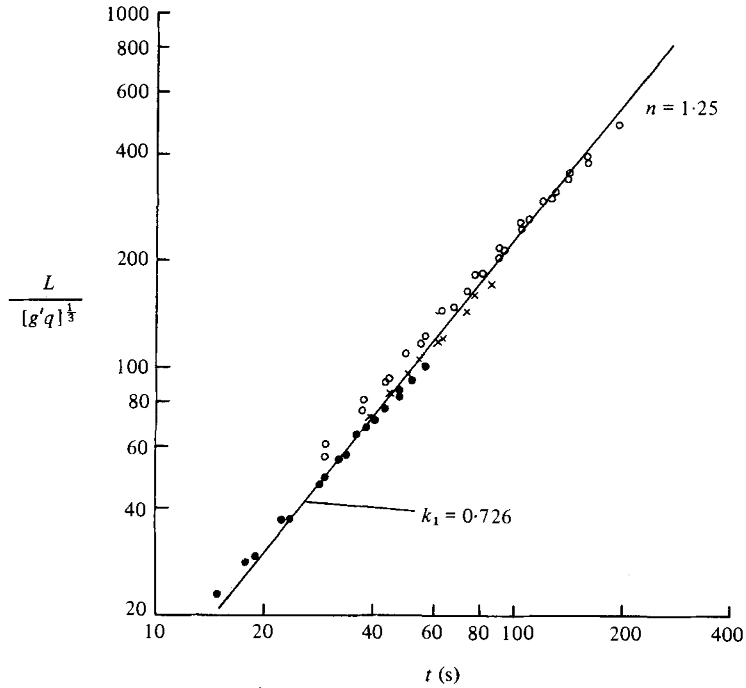

FIGURE 4. $L/[g'q]^{\frac{1}{3}}$ versus t for buoyancy–inertial currents: —○—, $\Delta\rho/\rho = 0{\cdot}0013$; —+—, 0·0038; —●—, 0·012; —×—, 0·063.

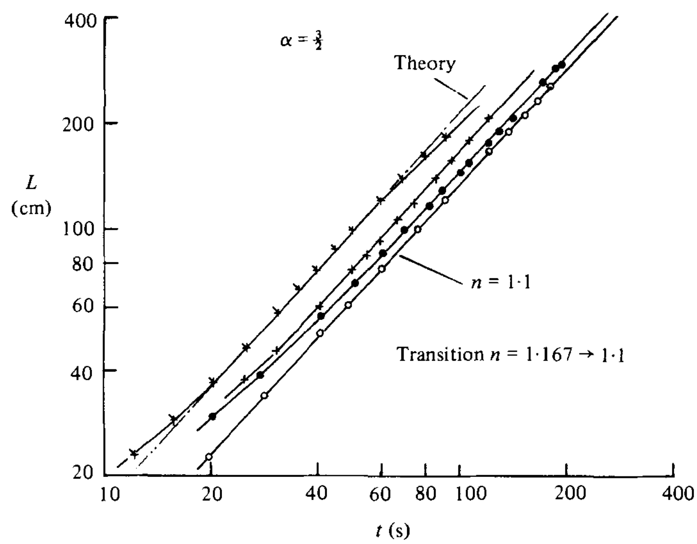

FIGURE 5. L versus t for $\alpha = \frac{3}{2}$ and $\Delta\rho/\rho = 0{\cdot}012$, showing agreement with Huppert (1982) except at very early times.

For values of α above the critical values we present two sets of results. For $\alpha = 3$ (figure 6) the initial viscous–buoyancy phase, in this case, is in close agreement with the theory of Huppert (1982), with the clearly observed transition to a lower slope close to that for an inertial–buoyancy balance (i.e. $n = 1{\cdot}667$) taking place at 195 s for the lower curve and 175 s for the upper. Here again the initial slope leading to the $n = 2{\cdot}0$ curve is very small (see §5). In figure 7 the results for $\alpha = 2$ are presented.

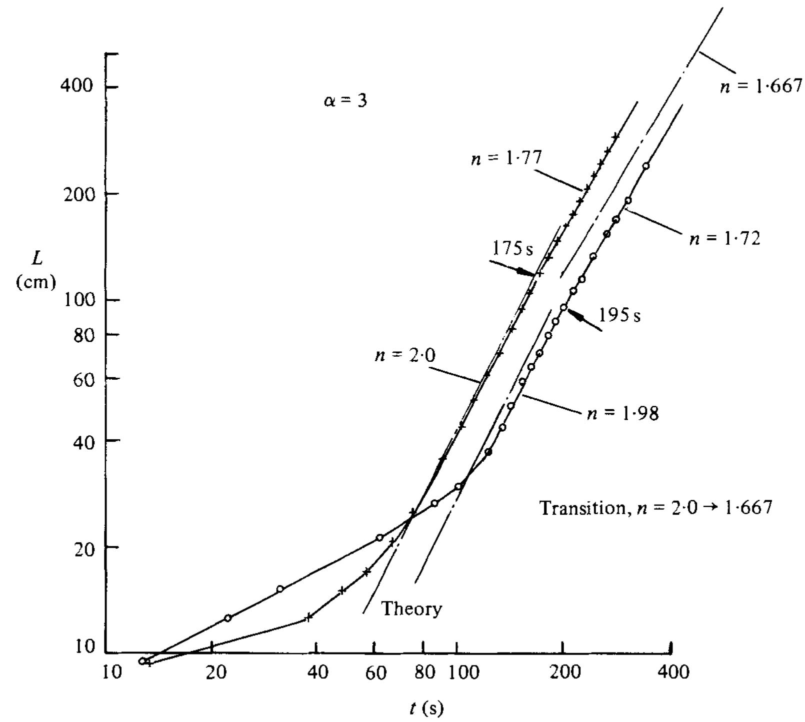

FIGURE 6. L versus t for $\alpha = 3$ and $\Delta\rho/\rho = 0{\cdot}012$. The agreement with theory for the initial buoyancy–viscous balance is good but the time at which transition to an inertial–buoyancy balance takes place is much smaller than estimated by using Huppert (1982).

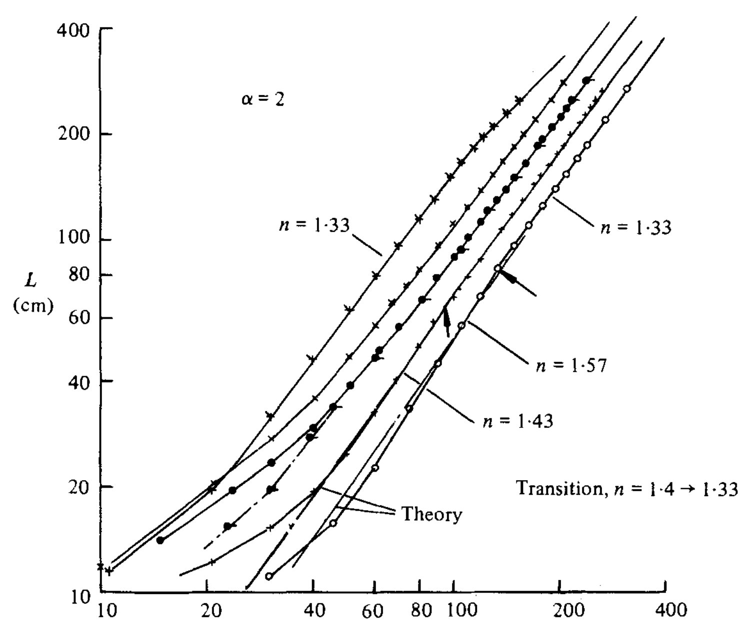

FIGURE 7. L versus t for $\alpha = 2$ and $\Delta\rho/\rho = 0{\cdot}012$. As in figure 6, transition, as indicated by arrows, takes place too soon, with the result that the inertially dominated region ($n = 1{\cdot}333$) is much lower than one would expect.

In the two cases for which a viscous–buoyancy balance exists initially the curves are both quite close to the theoretically predicted results, with transition to an inertial–buoyancy balance ($n = 1{\cdot}333$) taking place around 130 s in both cases. In all cases the agreement between the predicted slope of the inertial-buoyancy régime and the measurements is good but the values of $k_I(\alpha)$ show a considerable variation in contrast with the smaller, but nonetheless significant, scatter found for $\alpha = \frac{7}{4}$ (figure 4). There is a monotonic increase in $k_I(2)$, from 0·39 to 0·56, with q, for example, which suggests that perhaps mixing at the entry diffuser is important. To determine the validity of such a suggestion would require a detailed study of the flow field which is beyond the scope of the present experiments.

Finally we explain the change in slope over the last few points of the upper curves in both figures 5 and 7 as a consequence of the limitation of the pump used in the experiments. In these cases after 50 s its full capacity was being used so that for longer times the flow rate was constant. This information was eventually transmitted to the nose of the current, by waves on the interface between the two fluids, and it then responded to the new input conditions with a slope close to one, as would seem to be appropriate for an inertial–buoyancy current with constant inflow (but see the appendix for further comments).

5. Discussion

It appears that the arguments and accompanying theory put forward by Huppert (1982) (and to a lesser degree those in Didden & Maxworthy 1982) explain our present experiments quite well. Only two non-trivial discrepancies exist. The first concerns the initial slopes of some of the curves before they asymptote to their predicted values. In extreme cases, i.e. for $\alpha = 3$, these low values of the slope (usually less than $\frac{1}{2}$) persist for a long time, and in others, i.e. for $\alpha = \frac{3}{2}$ and 2, often overlap regions where flow-regime transitions might have taken place. These effects are almost entirely due to our experimental procedure, which required that we start each experiment with a small but nonetheless finite flow rate. Without some form of automated control system it was not possible to start the flow smoothly from zero, and so invariably the current was slightly longer than it should have been. Of course these small effects are greatly magnified on a log–log plot but they clearly have no effect on the final asymptotic behaviour, which, in every case but one, agreed to within 2 or 3 % of the theoretical values.

The second and more curious result concerns the variations in $k_I(\alpha)$ with q even when the temporal behaviour of the current indicates that the flow is self-similar and that the inertia–buoyancy balance represented by (2) is apparently valid. This question cannot be answered without recourse to the use of far more sophisticated equipment than that used here, since it will likely involve a direct measurement of the inertia and density distribution of the current and how these vary with the input parameters.

Finally we feel obliged to comment upon the initially surprising result that there are flows that can actually start by being viscously dominated. This at first might seem unlikely because all of the flows considered before these experiments and the theory of Huppert (i.e. these for which $\alpha = 0$ or 1) always start with an inertial–buoyancy behaviour and then undergo transition to a viscously dominated state. For values of $\alpha > \frac{7}{4}$, one can use the following physical arguments to show the logic of the result. One can divide the smoothly changing flow rate into small discrete steps, as in our experiment in fact. The initial small constant flow is such that it undergoes transition to a viscously controlled state very quickly. The next small increment is also quickly damped and so on until the flow rate is such that transition to a viscous state can no longer take place fast enough before a new increment of flow arrives. Finally transition to an inertial–buoyancy balance is complete.

For values of $\alpha < 1$ this type of argument also works well since now the flow rate is a decreasing function of time, having been infinite at $t = 0$, and one might expect an 'inertial' current to be formed initially. In the range $1 < \alpha < \frac{7}{4}$ again the initial flow rate is zero but it presumably increases fast enough for an inertial current to be formed at first and then to undergo transition to a viscously controlled flow.

The work reported here was performed while I was on sabbatical leave at the Institut für Hydromechanik at the Universität Karlsruhe as an Alexander von Humboldt U.S. Senior Scientist Awardee. I wish to thank Professor E. Naudascher for making the facilities of his institute available to me. I owe an especial debt to Professor Franz Durst, not only for applying for the award for me but also for making my stay at Karlsruhe such a rewarding experience, both scientifically and personally. Dr R. Ermshaus generously made the photographic equipment available, while Dr K. Faust collected all of the necessary equipment from the resources of the Institut Wasserbau III directed by Professor E. Plate. The manuscript was typed expertly by Frau R. Zschernitz and I thank her for dealing with my changes so patiently. Finally, I owe a great deal to my correspondence and discussions with Dr H. E. Huppert. He was a constant source of encouragement and valuable ideas and helped bring some order to much of my confusion over some of the more subtle aspects of this work.

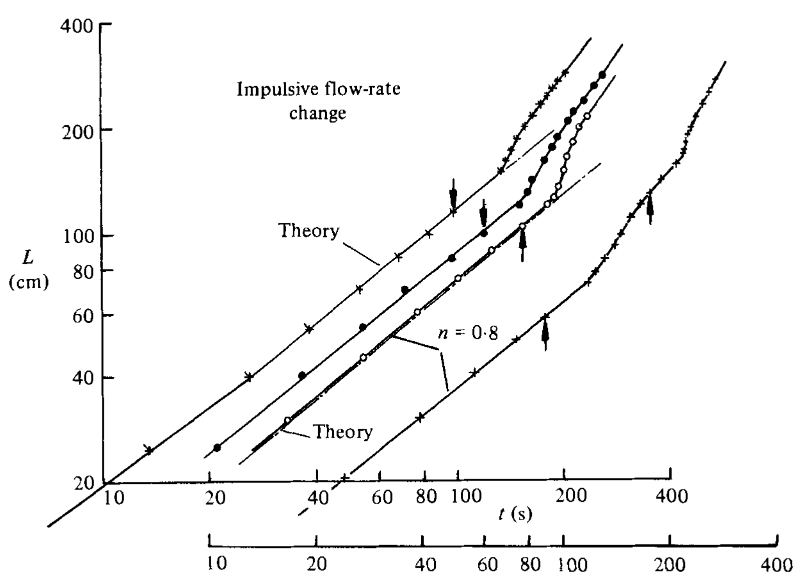

FIGURE 8. L versus t for $p = 0{\cdot}012$, with sudden large change in flow rate from one constant value to another at the times indicated by the arrows: —✱—, q changes from 0·90 to 2·77 cm²/s; —●—, 0·47 to 1·90 cm²/s; —○—, 0·40 to 2·70 cm²/s; —+—, 0·30 to 0·82 to 2·72 cm²/s. Note displaced abscissae for the lower curve.

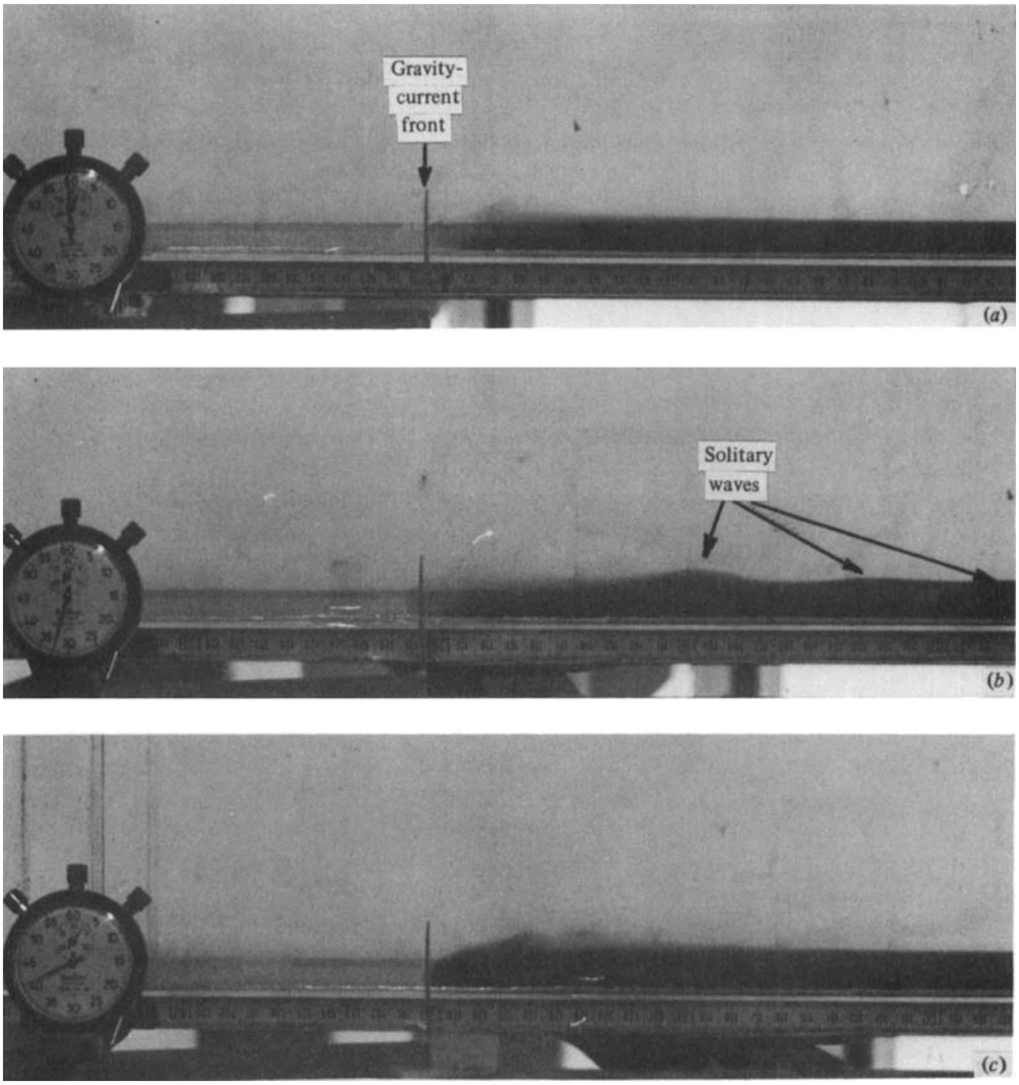

FIGURE 9. Photographs of the arrival of solitary waves at the nose of the gravity current after the flow rate was suddenly changed to a new value.

Appendix. Gravity currents with sudden flow-rate changes

As part of a series of initial experiments performed before those described in the main body of the paper, we explored the flow created when the flow rate into the current was suddenly changed from one constant value to another much higher value. Some of the results are shown in figure 8, where the initial slope is 0·8, as in Didden & Maxworthy (1982). At the times indicated by the arrows the flow rate was suddenly increased. A sequence of solitary waves, ordered by amplitude, was formed and these propagated on the interface between the two fluids until they reached the nose (see figure 9). At this point the latter became more bulbous and the slope of the curves increased rapidly to what appears to be a new constant slope considerably larger than unity in all cases. Since the current must eventually be indistinguishable from one that started with the final value of the flow, we presume that there is a suitable virtual origin that would show the final current on a trajectory with a slope of 0·8 or 1·0, depending on the force balance; unless, of course, the final asymptotic state had not yet been reached in the experiments. This result contrasts with that shown in the upper curves of figures 5 and 7, in which the anticipated slope of $n = 1$ was reached quickly when the flow rate became constant. In this latter case, of course, no solitary waves are possible, and this may account in some way for the different effects in the two cases. We have made no attempt to find this origin in the present cases, since the observation in which we had the greatest interest was to show the presence of solitary waves and indicate their importance as carriers of information in this flow field.

In the more general case, however, when the flow rate varies more or less smoothly it seems just as likely that this information will be carried by kinematic waves (Lighthill & Whitham 1955) as in flood waves on rivers, among many examples, and that dispersion will be reduced in importance.

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